Prime Suspect

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Editor's note: an email containing the following text was sent to Gareth, who spent six hours solving it with Richard. Gareth, a previous contributor to this magazine, wrote the solution up and donated it for publication.

> There are 2 integers n and m between 3 and 98 inclusive. Mr. S has been told their sum and Mr. P their product. The following truthful conversation occurs:
>
> P: I don't know n and m.
>
> S: I knew you didn't. Neither do I.
>
> P: Now I know them!
>
> S: Now I do, too!
>
> What are the values of n and m?
>
> P.S. I reckon you'll need a computer.

OK, let's make this sound mathematical. Let a *pair* be a set of two integers m and n such that $2 < m, n < 99$. Let us say that a pair whose product can be factorised into a different pair is *refactorisable* (we can also speak of their product as being a refactorisable product). To express a number as the sum of a pair is to *decompose* it.

The set of pairs is finite, and therefore so is the set of products; a little handle-turning could produce a list of refactorisable pairs (which we'll call set R). A computer could produce a complete list—but Richard and I didn't use that method. I'm not sure if that list would help you, anyway. Now, think about the problem...

Step One

Mr. P says "I don't know..."

Let's call the product $P = mn$, so P is a member of set R. That is, P is refactorisable.

Step Two

Mr. S says "I don't know"—so the sum, $S = m + n$, cannot be 6 or 7 or 195 or 196, because these sums would be uniquely decomposable.

Mr. S also says "I know Mr. P doesn't know". This means that *all* the decompositions of S must be refactorisable pairs. Let a number with the property that all decompositions are refactorisable pairs be called a *green* number, in recognition of my erstwhile colleague. The sum S is a green number.

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Now $7 < S < 195$. How many sums within this range have no non-refactorisable decompositions (that is, how many sums are green)? We shall work by a process of elimination, eliminating all values with *at least one* unique (that is, non-refactorisable) decomposition.

S must be less than 100. Since 97 is prime, $S > 99$ can be expressed as 97 plus some number between 3 and 98 inclusive. If this were the case, one of m and n might be 97, which gives a uniquely factorisable decomposition, and Mr. S couldn't be sure that P was refactorisable. This eliminates values up to $S = 97 + 98$, and we've already eliminated $S = 98 + 98$.

Similarly, S must be less than 56. Since 53 is the lowest prime greater than $98/2 = 49$, all S values between 56 and 100 can be decomposed as pairs containing 53, which are not refactorisable. So $7 < S < 56$.

Any even number can be expressed as a sum of two primes, so if S were even, Mr. S couldn't be sure m and n weren't both prime. Suppose S *is* even, and decompose it to a pair of primes. Since we have already restricted S below 98, both primes must be less than 98—and since $S = 4$ is not permitted, both primes must be odd, and so greater than or equal to 3. Both primes are therefore in the range 3–98 and are a valid, and uniquely factorisable, decomposition. Hence there are no even green numbers.

So S is odd, and must be the sum of an odd and an even. Because P is refactorisable, it must have at least three (not neccessarily distinct) prime factors, and because one of m and n is even, 2 must be a prime factor of P.

We can write a list of the remaining possible values of S, which will include every odd number between 9 and 55 inclusive. Are they all green? Let's now construct an algorithm to generate every possible decomposition of some S, and to refactorise the decomposition. If the algorithm fails for that S then that S is not green.

We know that we want an odd-even decomposition, which must be of the form $S = 2a + b$. Now $2a$ must be in the range 4 to $S - 3$ because of the limits on $2a$ and b. Can we always refactorise this to the pair $a, 2b$?

If $a = b$ and is prime then the refactorisation is not distinct. So no $S = 2a + a = 3a$ is green. All numbers of the form $3 \times \mathit{prime}$ can be struck off from our list of candidates.

If $2a = 4$ and b is prime, the pair is not refactorisable. So any S which can be expressed as $4 + \mathit{prime}$ is not green, and can be deleted from our list.

If b is non-prime (and $2a$ is still 4), it must have at least one prime factor no greater than its square root; since b is within the $2 < b < 99$ range, this prime factor is less than 10 and will multiply the 4 to a number no greater than 40—well within range. Since b is an odd nonprime, dividing out this prime factor must leave an odd remainder, hence the remainer is also greater than 2 and must be in range.

So if b is non-prime, we can then consider values of $2a$ between 6 and $S - 3$. For the possible values of S, a is always within range. Now $S < 57$ so $b = (S - 2a)$ is always less than 51. In fact, b, being odd, cannot exceed 49 and so $2b$ cannot exceed 98, and the refactorisation into $a, 2b$ is valid.

Hence the green numbers are all odd numbers between 7 and 55 inclusive which are not expressible as $3 \times \mathit{prime}$ or $4 + \mathit{prime}$.

This list can easily be constructed. We can identify the whole set of green numbers: 13, 19, 25, 29, 31, 37, 43, 49, 53 and 55.

Let any pair of numbers whose sum is green be said to be a green pair (pairs have many-to-one mappings to both the set of green sums and the set of refactorisable products).

Step Three

Mr. P says "Now I know!" P is a refactorisable number only one of whose factorisations is a green pair.

If a green pair can be refactorised to give another green pair, let us say it is *bright* green. If it cannot, let it be *pale* green. P's factorisation is neccessarily pale green.

If a product is refactorisable, it could be classified as being *dark* (no green factorisations, all the factorisations—of which there must be at least two—being 'grey' pairs,[3] pale (exactly one green factorisation—a pale green pair—and at least one grey factorisation), or bright (more than one green factorisation, all into bright green pairs, plus an indeterminate—zero, one or more than one—number of grey factorisations). Non-refactorisable products have a unique factorisation which cannot be a green pair by the definition of greenness.

Step Four

Mr. S says "Now I know, too!" S is a green number only one of whose decompositions is pale.

Step Five

What are m and n?

It turns out that the solution to step four is unique. All but one of the green numbers are multiply pale, that is, they have at least two pale decompositions (I don't know enough set theory to know if this neccessarily follows from the above). So we can go through our list of green numbers and eliminate all those with at least two pale decompositions.

We could take each green candidate and construct possible decompositions methodically: $3, (S-3)$; $4, (S-4)$; $5, (S-5)$; ... and test every possible refactorisation of each pair for greenness. We must test *every possible refactorisation* of a decomposition to make sure none of them is green, so that the decomposition is pale, but it is sufficient to find *two* pale decompositions to eliminate each candidate.

By way of a shortcut, we could be clever about the decompositions which we test, by looking for prime numbers p.

- Test (a): If the green candidate can be expressed as $(p + 2^z)$ then p will be odd and $z > 2$ ($z = 1$ is below limits, and if $z = 2$ the pair could not be green). There will be at least one refactorisation, to $2p$ and 2^{z-1}, and more if $z > 3$. But any possible refactorisation would be an even-even pair. Since all green numbers are odd, such a pair cannot possibly be green.
- Test (b): If the green candidate can be expressed as $(6 + p)$, the only possible refactorisation is to $3, 2p$. If $p > 26$ then the sum of the refactorisation will be too large to be green. Otherwise the refactorisation must be tested by inspection.
- Test (c): If the green candidate can be expressed as $(18 + p)$, refactorisations to $3p, 6$ and $3, 6p$ cannot give green pairs as the sum would be divisible by 3—and it can be

[3] In environmental jargon, the opposite of green is grey.

seen by inspection that none of the green numbers is a multiple of 3. The only other refactorisation is $9, 2p$ which must be tested by inspection.

Let's analyse all the green numbers for multiple paleness using these tests. All the following numbers can be eliminated.

- $55 = 47 + 8$ (pale by test 'a') $= 32 + 23$ (pale by test 'a')
- $53 = 37 + 16$ (pale by test 'a') $= 6 + 47$ (pale by test 'b' since $47 > 26$)
- $49 = 32 + 17$ (pale by test 'a') $= 41 + 8$ (pale by test 'a')
- $43 = 32 + 11$ (pale by test 'a') $= 6 + 37$ (pale by test 'b' since $37 > 26$)
- $37 = 32 + 5$ (pale by test 'a') $= 29 + 8$ (pale by test 'a')
- $31 = 23 + 8$ (pale by test 'a') $= 18 + 13$ (pale by test 'c' since it refactors to $9, 26$ whose sum is 35, which is grey)
- $25 = 8 + 17$ (pale by test 'a') $= 18 + 7$ (pale by test 'c' as it refactors to $9, 14$ and 23 is grey)
- $19 = 16 + 3$ (pale by test 'a') $= 11 + 8$ (pale by test 'a')
- $13 = 8 + 5$ (pale by test 'a') $= 6 + 7$ (pale by test 'b' since it refactors to $3, 14$ and 17 is grey)

So these three tests are *sufficient* to prove the multiple paleness of all green numbers other than 29. They are not sufficient to show that 29 has only one pale decomposition. We must test all possible decompositions of 29 for green refactorisations.

Decomposition	Refactorisation	Sum	Green?
3, 26	6, 13	19	yes
4, 25	20, 5	25	yes
5, 24	40, 3	43	yes
6, 23	46, 3	49	yes
7, 22	14, 11	25	yes
8, 21	24, 7	31	yes
9, 20	45, 4	49	yes
10, 19	38, 5	43	yes
11, 18	22, 9	31	yes
12, 17	51, 4	55	yes
13, 16	(pale by test 'a' above)		
14, 15	30, 7	37	yes

Hence, alone among the green numbers, 29 has only one pale decomposition. The values of m and n are 13 and 16. Mr. P's product is 208. Mr. S's sum is 29. Problem solved.

Despite the hint that "you will probably need a computer", it was done in practice using only two neural networks running wetware and a calculator. The solution above (greatly refined by virtue of being understood now) can be understood without use even of a calculator.